Given:

$\pi_{\Theta}(\theta)$

Prior distribution: $\pi_{\Theta}(1) = \pi_{\Theta}(3) = 0.5$

conditional distribution $f_{X|\Theta}(x|\theta) = \frac{\theta}{(x+\theta)^2}$, $0 < x < \infty$

For a given policy observe: $X_1 = 5$

Want:

Predictive probability $Pr(X_2 > 8 \mid X_1 = 5) = ?$

$$f_{X|\Theta}(5|1) = \frac{1}{(5+1)^2} = \frac{1}{36}$$

$$f_{X|\Theta}(5|3) = \frac{3}{(5+3)^2} = \frac{3}{64}$$

joint probabilities $f_{X,\Theta}(5,1) = f_{X|\Theta}(5|1) \cdot \pi_{\Theta}(1)$

$$= \frac{1}{36} \times 0.5 = \frac{1}{72}$$

$$f_{X,\Theta}(5,3) = f_{X|\Theta}(5|3) \cdot \pi_{\Theta}(3) = \frac{3}{64} \times 0.5 = \frac{3}{128}$$

marginal probability

$$f_X(5) = \sum_{\theta} f_{X|\Theta}(5|\theta)\, \pi_{\Theta}(\theta) = f_{X|\Theta}(5|1)\, \pi_{\Theta}(1) + f_{X|\Theta}(5|3)\, \pi_{\Theta}(3)$$

$$= \frac{1}{36} \times 0.5 + \frac{3}{64} \times 0.5 = 0.0373$$

posterior distribution of θ

$$\pi_{\Theta|X}(1|5) = \frac{f_{X,\Theta}(5,1)}{f_X(5)} = \frac{f_{X|\Theta}(5|1)\,\pi_\Theta(1)}{f_X(5)}$$

$$= \frac{1}{36} \times 0.5 / 0.0373 = \frac{16}{43} = 0.3721$$

$$\pi_{\Theta|X}(3|5) = \frac{f_{X;\Theta}(5,3)}{f_X(5)} = \frac{f_{X|\Theta}(5|3)\,\pi_\Theta(3)}{f_X(5)}$$

$$= \frac{3}{64} \times 0.5 / 0.0373 = \frac{27}{43} = 0.6284$$

Conditional probability that loss exceeds 8

$$Pr(X>8|\theta) = \int_8^\infty f_{X|\Theta}(x|\theta)\,dx = \int_8^\infty f(x|\theta)\,dx$$

$$= \int_8^\infty \frac{\theta}{(x+\theta)^2}\,dx = \theta \cdot \int_8^\infty \frac{1}{(x+\theta)^2}\,dx = -\frac{\theta}{x+\theta}\Big|_8^\infty$$

$$= \cancel{\theta} = 0 - \left(-\frac{\theta}{8+\theta}\right) = \frac{\theta}{8+\theta}$$

$$Pr(X>8|1) = \frac{1}{8+1} = \frac{1}{9}$$

$$Pr(X>8|3) = \frac{3}{8+3} = \frac{3}{11}$$

Predictive probability that

$$Pr(X_2>8|X_1=5) = \sum_\theta Pr(X>8|\theta)\,\pi_{\Theta|X}(\theta|5)$$

$$= Pr(X>8|1)\,\pi_{\Theta|X}(1|5) + Pr(X>8|3)\,\pi_{\Theta|X}(3|5)$$

$$= \frac{1}{9} \times \frac{16}{43} + \frac{3}{11} \times \frac{27}{43} = 0.2126$$

$$= \frac{1}{9} \times 0.3721 + \frac{3}{11} \times 0.6284 = 0.2126$$