Question #130
Key: E

$$E(W)=\frac{1}{4}\int_0^4\sum_{i=0}^{\infty}2^i\Pr\left(N=i\middle|\lambda\right)d\lambda \qquad \left[\frac{1}{4}\text{ is the density of }\lambda\text{ on }[0,4].\right]$$

$$=\frac{1}{4}\int_0^4 P\left(2\middle|\lambda\right)d\lambda \qquad [\text{see note}]$$

$$=\frac{1}{4}\int_0^4 e^{\lambda(2-1)}d\lambda \qquad [\text{using formula from tables for the pgf of the Poisson}]$$

$$=\frac{1}{4}e^{\lambda}\Big|_0^4=\frac{1}{4}\left(e^4-1\right)$$

$$=13.4$$

Note: the probability generating function (pgf) is $P(Z)=\sum_{k=0}^{\infty} p_k Z^k$ so the integrand is $P(2)$, or in this case $P\left(2\middle|\lambda\right)$ since λ is not known.

Alternatively,

$$E(W)=\frac{1}{4}\int_0^4\sum_{i=0}^{\infty}2^i\Pr\left(N=i\middle|\lambda\right)d\lambda$$

$$=\frac{1}{4}\int_0^4\sum_{i=0}^{\infty}\frac{2^i e^{-\lambda}\lambda^i}{i!}d\lambda$$

$$=\frac{1}{4}\int_0^4\sum_{i=0}^{\infty}\frac{e^{-\lambda}(2\lambda)^i}{i!}d\lambda$$

We know $\sum_{i=0}^{\infty}\frac{e^{-2\lambda}(2\lambda)^i}{i!}=1$ since $\frac{e^{-2\lambda}(2\lambda)^i}{i!}$ is $f(i)$ for a Poisson with mean $Z\lambda$

so $\sum_{i=0}^{\infty}\frac{e^{-\lambda}(2\lambda)^i}{i!}=\frac{e^{-\lambda}}{e^{-2\lambda}}=e^{\lambda}$

Thus $E(W)=\frac{1}{4}\int_0^4 e^{\lambda}d\lambda$

$$=\frac{1}{4}e^{\lambda}\Big|_0^4=\frac{1}{4}\left(e^4-1\right)$$

$$=13.4$$