**Solution to (16)** Answer (B)

It follows from (20.30) in Proposition 20.3 that

$$F_{t,T}^{P}[S(T)^x] = S(t)^x \exp\{[-r + x(r - \delta) + \tfrac{1}{2}x(x-1)\sigma^2](T-t)\},$$

which equals $S(t)^x$ if and only if

$$-r + x(r - \delta) + \tfrac{1}{2}x(x-1)\sigma^2 = 0.$$

This is a quadratic equation of $x$. With $\delta = 0$, the quadratic equation becomes

$$\begin{aligned} 0 &= -r + xr + \tfrac{1}{2}x(x-1)\sigma^2 \\ &= (x-1)(\tfrac{1}{2}\sigma^2 x + r). \end{aligned}$$

Thus, the solutions are 1 and $-2r/\sigma^2 = -2(4\%)/(20\%)^2 = -2$, which is (B).

**Remarks**:

(i) McDonald (2006, Section 20.7) has provided three derivations for (20.30). Here is another derivation. Define $Y = \ln[S(T)/S(t)]$. Then,

$$\begin{aligned} F_{t,T}^{P}[S(T)^x] &= E_t^*[e^{-r(T-t)}\, S(T)^x] && \because \text{ Prepaid forward price} \\ &= E_t^*[e^{-r(T-t)}\, (S(t)e^Y)^x] && \because \text{ Definition of } Y \\ &= e^{-r(T-t)}\, S(t)^x\, E_t^*[e^{xY}]. && \because \text{ The value of } S(t) \text{ is not random at time } t \end{aligned}$$

The problem is to find $x$ such that $e^{-r(T-t)} E_t^*[e^{xY}] = 1$. To evaluate the expectation $E_t^*[e^{xY}]$, note that, under the risk-neutral probability measure, $Y$ is a normal random variable with mean $(r - \delta - \tfrac{1}{2}\sigma^2)(T-t)$ and variance $\sigma^2(T-t)$. Thus, by the moment-generating function formula for a normal random variable or by formula (18.13) in McDonald (2006),

$$E_t^*[e^{xY}] = \exp[x(r - \delta - \tfrac{1}{2}\sigma^2)(T-t) + \tfrac{1}{2}x^2\sigma^2(T-t)].$$

Hence the equation $e^{-r(T-t)} E_t^*[e^{xY}] = 1$ becomes

$$-r(T-t) + x(r - \delta - \tfrac{1}{2}\sigma^2)(T-t) + \tfrac{1}{2}x^2\sigma^2(T-t) = 0,$$

which yields the same quadratic equation of $x$ as above.

(ii) The two solutions of the quadratic equation,

$$\tfrac{1}{2}\sigma^2 x^2 + (r - \delta - \tfrac{1}{2}\sigma^2)x - r = 0,$$

are $x = h_1$ and $x = h_2$ in Section 12.6 of McDonald (2006). A reason for this "coincidence" is that $h_1$ and $h_2$ are the values of $x$ for which the stochastic process $\{e^{-rt}\, S(t)^x\}$ becomes a *martingale*. Martingales are mentioned on page 651 of McDonald (2006).

(iii) Before time $T$, the contingent claim does not pay anything. Thus, the prepaid forward price at time $t$ is in fact the time-$t$ price of the contingent claim.