**Solution to (36)** Answer: (E)

This problem is a special case of Exercise 21.2 where $\gamma = 0$, $\delta = 0$, and $a = -k/\sigma^2$. Also, it is a variation of #16 in this set of Sample Questions and Solutions.

**First Solution**: We are given that the time-$t$ price of the derivative security is

$$V(S(t), t) = [S(t)]^a,$$

where $a$ is a negative constant. The function $V(s, t)$ must satisfy the Black-Scholes partial differential equation (21.11)

$$\frac{\partial V}{\partial t} + (r - \delta)s\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} = rV$$

(where $\delta = 0$ because the stock does not pay dividends). Since $V(s, t) = s^a$, we have $V_t = 0$, $V_s = as^{a-1}$, and $V_{ss} = a(a-1)s^{a-2}$. Thus, the B-S PDE becomes

$$0 + (r - \delta)s\left(as^{a-1}\right) + \frac{1}{2}\sigma^2 s^2\left(a(a-1)s^{a-2}\right) = rs^a,$$

or

$$(r - \delta)a + \frac{1}{2}\sigma^2 a(a-1) = r,$$

which is a quadratic equation of $a$. With $\delta = 0$, one obvious solution is $a = 1$ (which is not negative). The other solutions is $a = -2r/\sigma^2$. Consequently, $k = 2r = 2\times0.04 = 0.08$.

**Second Solution:**

Let $V[S(t), t]$ denote the time-$t$ price of a derivative security that does not pay dividends. Then, for $t \le T$,

$$V[S(t), t] = F^P_{t,T}(V[S(T), T]).$$

In particular,

$$V[S(0), 0] = F^P_{0,T}(V[S(T), T]).$$

Because in this problem are given that $V[S(t), t] = [S(t)]^a$, where $a = -k/\sigma^2$, the equation above is

$$[S(0)]^a = F^P_{0,T}([S(T)]^a)$$

$$= e^{-rT}[S(0)]^a \exp\{[a(r - \delta) + ½a(a - 1)\sigma^2]T\}$$

by (20.30). Hence we have the quadratic equation

$$0 = -r + a(r - \delta) + ½a(a - 1)\sigma^2,$$

which is the same as the one above.

**Third Solution:**

For simplicity, write $-k/\sigma^2$ as $a$ and $[S(t)]^a$ as $S^a(t)$. Differentiating $S^a(t)$ by means of Itô's Lemma yields

$$\mathrm{d}S^a(t) = aS^{a-1}(t)\mathrm{d}S(t) + \tfrac{1}{2}a(a-1)S^{a-2}(t)[\mathrm{d}S(t)]^2 + 0\mathrm{d}t$$
$$= aS^{a-1}(t)[(\alpha - \delta)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}Z(t)] + \tfrac{1}{2}a(a-1)S^{a-2}(t)\sigma^2 S^2(t)\mathrm{d}t$$
$$= \left[a(\alpha-\delta) + \tfrac{1}{2}a(a-1)\right]S^a(t)\mathrm{d}t + a\sigma S^a(t)\mathrm{d}Z(t).$$

Hence,

$$\frac{\mathrm{d}S^a(t)}{S^a(t)} = \left[a(\alpha-\delta) + \tfrac{1}{2}a(a-1)\sigma^2\right]\mathrm{d}t + a\sigma \mathrm{d}Z(t),$$

which is (20.32) in McDonald (2006). Because $S^a(t)$ is the *price* of a (tradable) security, the no-arbitrage argument in Section 20.4 "The Sharpe Ratio" shows that

$$\frac{\left[a(\alpha-\delta) + \tfrac{1}{2}a(a-1)\sigma^2\right] - r}{a\sigma} = \frac{\alpha - r}{\sigma},$$

which gives the same quadratic equation of $a$ as above.

**Remarks**:

(i) The denominator in the LHS of the last equation is $a\sigma$, not its absolute value.

(ii) If $\delta > 0$, the solutions of the quadratic equation are $a = h_1 > 1$ and $a = h_2 < 0$ as defined in Section 12.6 of McDonald (2006). Section 12.6 is not currently in the syllabus of Exam MFE/3F.

(iii) For those who know *martingale* theory, the second solution is equivalent to seeking $a$ such that, under the risk-neutral probability measure, the stochastic process $\{e^{-rt}[S(t)]^a;\ t \geq 0\}$ is a martingale. There are two such martingales.

(iv) The second solution requires formula (20.30) or (20.31). To derive (20.31), we can use the fact $F_{0,T}(S^a) = \mathrm{E}^*[S^a(T)]$. Now,

$$S(t) = S(0)\exp[(\alpha - \delta - \tfrac{1}{2}\sigma^2)t + \sigma Z(t)] = S(0)\exp[(r - \delta - \tfrac{1}{2}\sigma^2)t + \sigma\tilde{Z}(t)],$$

where $\tilde{Z}(t) = Z(t) + [(\alpha - r)/\sigma]t$. Under the risk-neutral probability measure, $\{\tilde{Z}(t)\}$ is a standard Brownian motion. Thus,

$$\mathrm{E}^*[S^a(T)] = S^a(0)\exp[a(r - \delta - \tfrac{1}{2}\sigma^2)T + \tfrac{1}{2}(a\sigma)^2T],$$

yielding (20.31). The last equation is the same as (20.35) in McDonald (2006).

(v) If the derivative security pays dividends, then its price, $V$, does not satisfy the partial differential equation (21.11). If the dividend payment between time $t$ and time $t + \mathrm{d}t$ is $\Gamma(t)\mathrm{d}t$, then the Black-Scholes equation (21.31) will need to be modified as

$$\mathrm{E}_t^*[\mathrm{d}V + \Gamma(t)\mathrm{d}t] = V \times (r\mathrm{d}t).$$

See also Exercise 21.10 on page 700.