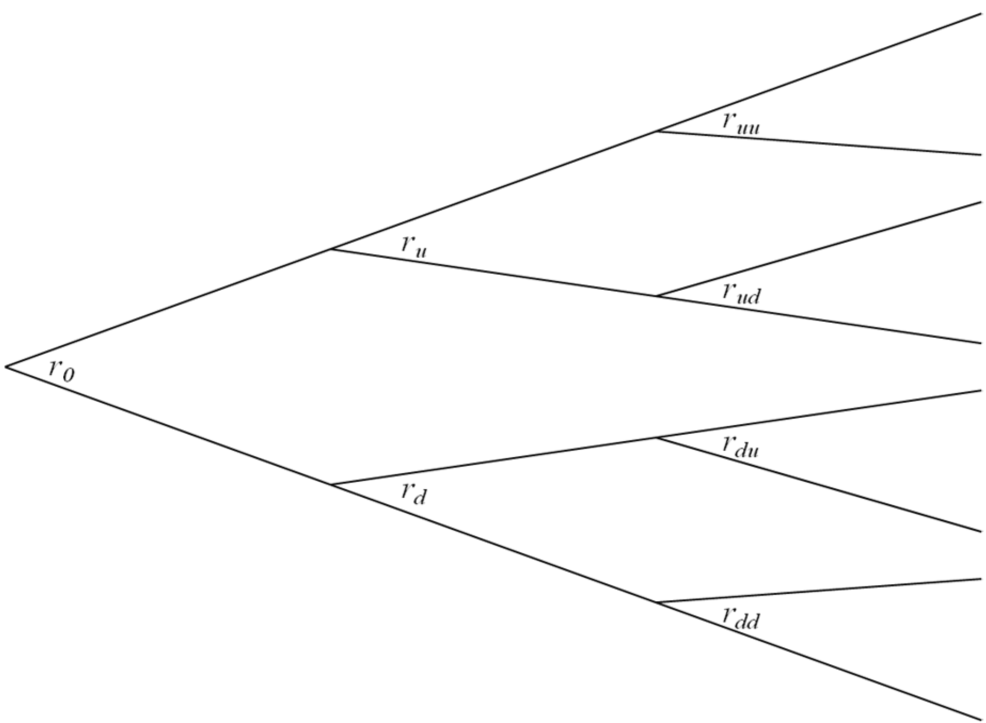
Solution to (76) Answer: (D)

A related problem is #15 in this set of sample questions and solutions.

Caps are usually defined so that the initial rate, r_0, even if it is greater than the cap rate, does not lead to a payoff, i.e., the year-1 caplet is disregarded. In any case, the year-1 caplet in this problem has zero value because r_0 is lower than the cap rate.

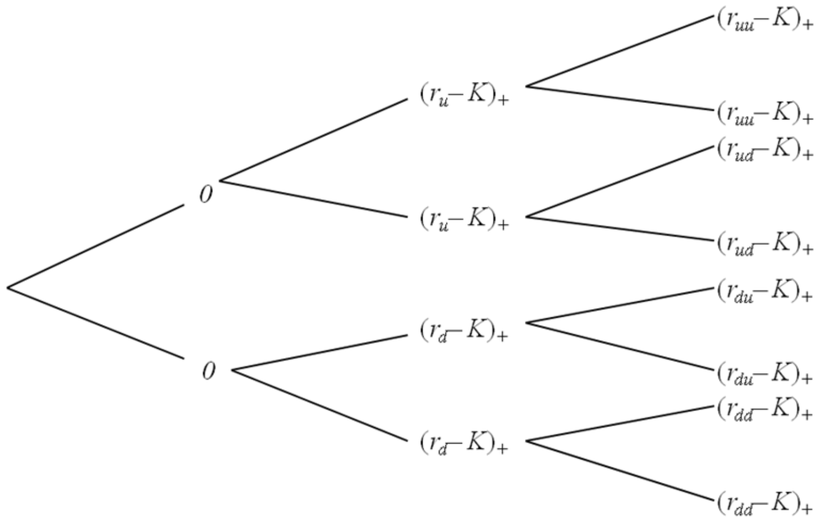
Since a 3-year cap is the sum of a year-2 caplet and a year-3 caplet, one way to price a 3-year cap is to price each of the two caplets and then add up the two prices. However, because the payoffs or cashflows of a cap are not path-dependent, the method of *backward induction* can be applied, which is what we do next.

It seems more instructive if we do not assume that the binomial tree is recombining, i.e., we do not assume $r_{ud} = r_{du}$. Thus we have the following three-period (three-year) interest rate tree.

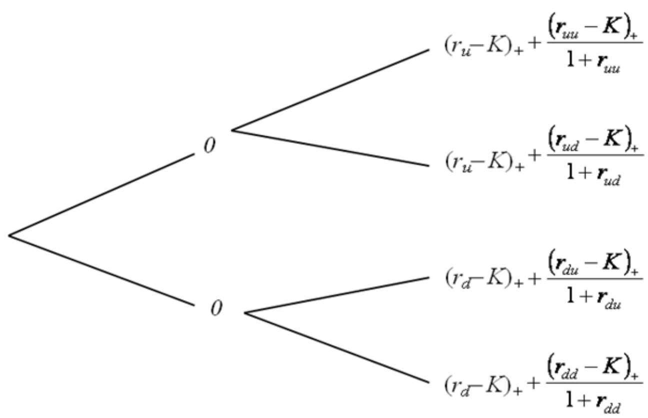


We also do not assume the risk-neutral probabilities to be ½ and ½. We use p^* to denote the risk-neutral probability of an up move, and q^* the probability of a down move.

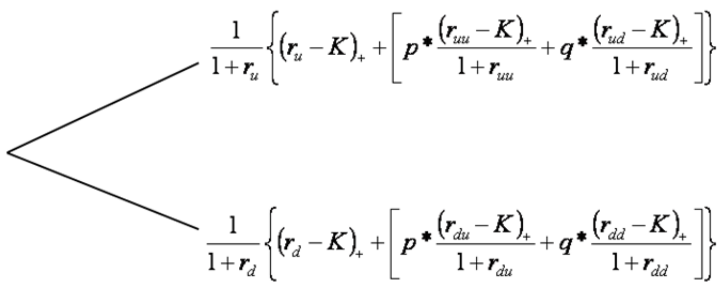
In the next diagram, we show the payoffs or cashflows of a 3-year interest-rate cap for notional amount 1 and cap rate K. Here, $(r - K)_+$ means $\max(0, r - K)$.



Discounting and averaging the cashflows at time 3 back to time 2:



Moving back from time 2 back to time 1:



Finally, we have the time-0 price of the 3-year interest-rate cap for notional amount 1 and cap rate K:

$$\frac{1}{1+r_0}\left\{p*\frac{1}{1+r_u}\left[(r_u-K)_+ + p*\frac{(r_{uu}-K)_+}{1+r_{uu}} + q*\frac{(r_{ud}-K)_+}{1+r_{ud}}\right]\right.$$
$$\left. + q*\frac{1}{1+r_d}\left[(r_d-K)_+ + p*\frac{(r_{du}-K)_+}{1+r_{du}} + q*\frac{(r_{dd}-K)_+}{1+r_{dd}}\right]\right\}. \qquad (1)$$

As we mentioned earlier, the price of a cap can also be calculated as the sum of caplet prices. The time-0 price of a year-2 caplet is

$$\frac{1}{1+r_0}\left[p^*\frac{(r_u-K)_+}{1+r_u} + q^*\frac{(r_d-K)_+}{1+r_d}\right].$$

The time-0 price of a year-3 caplet is

$$\frac{1}{1+r_0}\left\{p*\frac{1}{1+r_u}\left[p*\frac{(r_{uu}-K)_+}{1+r_{uu}} + q*\frac{(r_{ud}-K)_+}{1+r_{ud}}\right] + q*\frac{1}{1+r_d}\left[p*\frac{(r_{du}-K)_+}{1+r_{du}} + q*\frac{(r_{dd}-K)_+}{1+r_{dd}}\right]\right\}. \qquad (2)$$

It is easy to check that the sum of these two caplet pricing formulas is the same as expression (1).

Rewriting expression (2) as

$$(p*)^2\frac{(r_{uu}-K)_+}{(1+r_0)(1+r_u)(1+r_{uu})} + p*q*\frac{(r_{ud}-K)_+}{(1+r_0)(1+r_u)(1+r_{ud})}$$
$$+q*p*\frac{(r_{du}-K)_+}{(1+r_0)(1+r_d)(1+r_{du})} + (q*)^2\frac{(r_{du}-K)_+}{(1+r_0)(1+r_d)(1+r_{dd})}$$

shows the path-by-path nature of the year-3 caplet price.

A Black-Derman-Toy interest rate tree is a recombining tree (hence $r_{ud} = r_{du}$) with $p* = q* = ½$. Expression (1) simplifies as

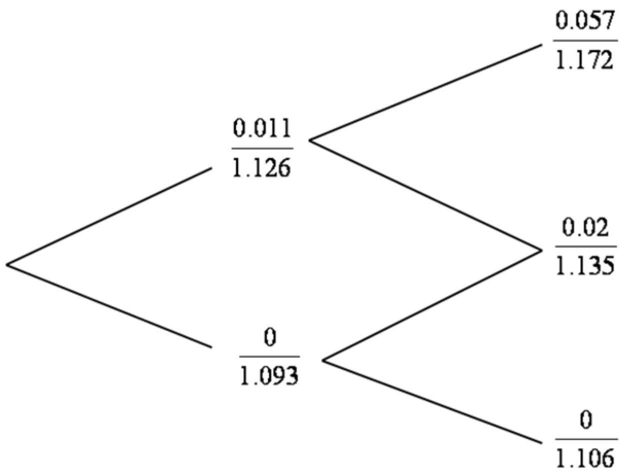
$$\frac{1}{2}\frac{1}{1+r_0}\left\{\frac{1}{1+r_u}\left[(r_u-K)_+ + \frac{1}{2}\left(\frac{(r_{uu}-K)_+}{1+r_{uu}} + \frac{(r_{ud}-K)_+}{1+r_{ud}}\right)\right]\right.$$
$$\left. + \frac{1}{1+r_d}\left[(r_d-K)_+ + \frac{1}{2}\left(\frac{(r_{ud}-K)_+}{1+r_{ud}} + \frac{(r_{dd}-K)_+}{1+r_{dd}}\right)\right]\right\}. \qquad (3)$$

In this problem, the value of r_{ud} is not given. In each period of a B-D-T model, the interest rates in different states are terms of a geometric progression. Thus, we have

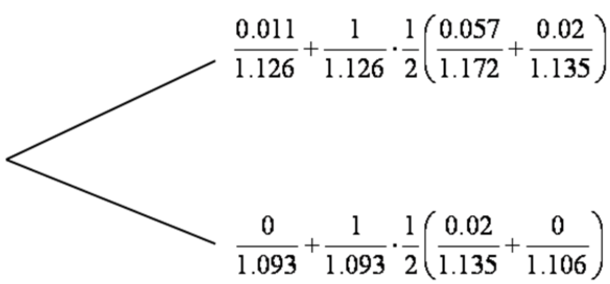
$$0.172/r_{ud} = r_{ud}/0.106,$$

from which we obtain $r_{ud} = 0.135$. With this value, we can price the cap using (3).

Instead of using (3), we now solve the problem directly. As in Figure 24.9 on page 806 of McDonald (2006), we first discount each cap payment to the beginning of the payment year. The following tree is for notational amount of 1 (and $K = 0.115$).



Discounting and averaging the cashflows at time 2 back to time 1:



Thus the time-0 price of the cap is

$$\frac{1}{1.09} \cdot \frac{1}{2}\left\{\frac{1}{1.126}\left[0.011 + \frac{1}{2}\left(\frac{0.057}{1.172} + \frac{0.02}{1.135}\right)\right] + \frac{1}{1.093}\left[0 + \frac{1}{2}\left(\frac{0.02}{1.135} + \frac{0}{1.106}\right)\right]\right\}$$
$$= \frac{1}{1.09} \cdot \frac{1}{2}\left\{\frac{1}{1.126} \times 0.044128 + \frac{1}{1.093} \times 0.008811\right\}$$
$$= 0.02167474.$$

Answer (D) is correct, because the notation amount is 10,000.

Remark: The prices of the two caplets for notional amount 10,000 and $K = 0.115$ are 44.81 and 171.94. The sum of these two prices is 216.75.