Question #11
Answer: E

Let NS denote non-smokers and S denote smokers.

The shortest solution is based on the conditional variance formula

$$\text{Var}(X) = E\left(\text{Var}(X|Y)\right) + \text{Var}\left(E(X|Y)\right)$$

Let *Y* = 1 if smoker; *Y* = 0 if non-smoker

$$E\left(\bar{a}_{\overline{T|}} \middle| Y=1\right) = \bar{a}_x^S = \frac{1-\bar{A}_x^S}{\delta}$$

$$= \frac{1-0.444}{0.1} = 5.56$$

Similarly $E\left(\bar{a}_{\overline{T|}} \middle| Y=0\right) = \dfrac{1-0.286}{0.1} = 7.14$

$$E\left(E\left(\bar{a}_{\overline{T|}} \middle| Y\right)\right) = E\left(E\left(\bar{a}_{\overline{T|}} \middle| 0\right)\right) \times \text{Prob}(\text{Y}=0) + E\left(E\left(\bar{a}_{\overline{T|}} \middle| 1\right)\right) \times \text{Prob}(\text{Y}=1)$$

$$= (7.14)(0.70) + (5.56)(0.30)$$

$$= 6.67$$

$$E\left[\left(E\left(\bar{a}_{\overline{T}|}|Y\right)\right)^2\right]=\left(7.14^2\right)(0.70)+\left(5.56^2\right)(0.30)$$
$$=44.96$$
$$\text{Var}\left(E\left(\bar{a}_{\overline{T}|}|Y\right)\right)=44.96-6.67^2=0.47$$
$$E\left(\text{Var}\left(\bar{a}_{\overline{T}|}|Y\right)\right)=(8.503)(0.70)+(8.818)(0.30)$$
$$=8.60$$
$$\text{Var}\left(\bar{a}_{\overline{T}|}\right)=8.60+0.47=9.07$$

Alternatively, here is a solution based on
$\text{Var}(Y)=E\left(Y^2\right)-\left[E(Y)\right]^2$, a formula for the variance of any random variable.
This can be
transformed into $E\left(Y^2\right)=\text{Var}(Y)+\left[E(Y)\right]^2$ which we will use in its conditional form

$$E\left(\left(\bar{a}_{\overline{T}|}\right)^2|\text{NS}\right)=\text{Var}\left(\bar{a}_{\overline{T}|}|\text{NS}\right)+\left[E\left(\bar{a}_{\overline{T}|}|\text{NS}\right)\right]^2$$

$$\text{Var}\left[\bar{a}_{\overline{T}|}\right]=E\left[\left(\bar{a}_{\overline{T}|}\right)^2\right]-\left(E\left[\bar{a}_{\overline{T}|}\right]\right)^2$$
$$E\left[\bar{a}_{\overline{T}|}\right]=E\left[\bar{a}_{\overline{T}|}|\text{S}\right]\times\text{Prob}[\text{S}]+E\left[\bar{a}_{\overline{T}|}|\text{NS}\right]\times\text{Prob}[\text{NS}]$$
$$=0.30\bar{a}_x^{\text{S}}+0.70\bar{a}_x^{\text{NS}}$$
$$=\frac{0.30\left(1-\bar{A}_x^{\text{S}}\right)}{0.1}+\frac{0.70\left(1-\bar{A}_x^{\text{NS}}\right)}{0.1}$$
$$=\frac{0.30(1-0.444)+0.70(1-0.286)}{0.1}=(0.30)(5.56)+(0.70)(7.14)$$
$$=1.67+5.00=6.67$$

$$E\left[\left(\bar{a}_{\overline{T}|}\right)^2\right]=E\left[\bar{a}_{\overline{T}|}^2|\text{S}\right]\times\text{Prob}[\text{S}]+E\left[\bar{a}_{\overline{T}|}^2|\text{NS}\right]\times\text{Prob}[\text{NS}]$$
$$=0.30\left(\text{Var}\left(\bar{a}_{\overline{T}|}|\text{S}\right)+\left(E\left[\bar{a}_{\overline{T}|}|\text{S}\right]\right)^2\right)$$
$$+0.70\left(\text{Var}\left(\bar{a}_{\overline{T}|}|\text{NS}\right)+E\left(\bar{a}_{\overline{T}|}|\text{NS}\right)^2\right)$$
$$=0.30\left[8.818+(5.56)^2\right]+0.70\left[8.503+(7.14)^2\right]$$
11.919 + 41.638 = 53.557

$$\text{Var}\left[\bar{a}_{\overline{T}|}\right]=53.557-(6.67)^2=9.1$$

Alternatively, here is a solution based on $\bar{a}_{\overline{T|}} = \dfrac{1-v^T}{\delta}$

$$
\begin{aligned}
\operatorname{Var}\left(\bar{a}_{\overline{T|}}\right) &= \operatorname{Var}\left(\frac{1}{\delta}-\frac{v^T}{\delta}\right) \\
&= \operatorname{Var}\left(\frac{-v^T}{\delta}\right) \text{ since } \operatorname{Var}(X+\text{constant}) = \operatorname{Var}(X) \\
&= \frac{\operatorname{Var}\left(v^T\right)}{\delta^2} \text{ since } \operatorname{Var}(\text{constant}\times X) = \text{constant}^2 \times \operatorname{Var}(X) \\
&= \frac{{}^2\bar{A}_x - \left(\bar{A}_x\right)^2}{\delta^2} \text{ which is Bowers formula 5.2.9}
\end{aligned}
$$

This could be transformed into ${}^2A_x = \delta^2 \operatorname{Var}\left(\bar{a}_{\overline{T|}}\right) + \bar{A}_x^2$, which we will use to get ${}^2A_x{}^{\text{NS}}$ and ${}^2A_x{}^{\text{S}}$.

$$
\begin{aligned}
{}^2A_x &= E\left[v^{2T}\right] \\
&= E\left[v^{2T} \mid \text{NS}\right] \times \text{Prob}(\text{NS}) + E\left[v^{2T} \mid \text{S}\right] \times \text{Prob}(\text{S}) \\
&= \left[\delta^2 \operatorname{Var}\left(\bar{a}_{\overline{T|}} \mid \text{NS}\right) + \left(\bar{A}_x{}^{\text{NS}}\right)^2\right] \times \text{Prob}(\text{NS}) \\
&\quad + \left[\delta^2 \operatorname{Var}\left(\bar{a}_{\overline{T|}} \mid \text{S}\right) + \left(\bar{A}_x{}^{\text{S}}\right)^2\right] \times \text{Prob}(\text{S}) \\
&= \left[(0.01)(8.503) + 0.286^2\right] \times 0.70 \\
&\quad + \left[(0.01)(8.818) + 0.444^2\right] \times 0.30 \\
&= (0.16683)(0.70) + (0.28532)(0.30) \\
&= 0.20238
\end{aligned}
$$

$$
\begin{aligned}
\bar{A}_x &= E\left[v^T\right] \\
&= E\left[v^T \mid \text{NS}\right] \times \text{Prob}(\text{NS}) + E\left[v^T \mid \text{S}\right] \times \text{Prob}(\text{S}) \\
&= (0.286)(0.70) + (0.444)(0.30) \\
&= 0.3334
\end{aligned}
$$

$$\mathrm{Var}\left(\bar{a}_{\overline{T}|}\right) = \frac{{}^{2}\bar{A}_x - \left(\bar{A}_x\right)^2}{\delta^2}$$

$$= \frac{0.20238 - 0.3334^2}{0.01} = 9.12$$