

Problem 115

$$f(x, y) = 2x \quad 0 < x < 1; x < y < x+1$$

$$V(Y|X=x) = E(Y^2|X=x) - [E(Y|X=x)]^2$$

$$f(Y|X=x) = \frac{f(x, y)}{f(x)}$$

$$\begin{aligned} f(x) &= \int_x^{x+1} 2x \cdot dy \\ &= 2xy \Big|_x^{x+1} = 2x(x+1-x) = 2x \Rightarrow f(Y|X=x) = \frac{2x}{2x} = 1 \quad x < y < x+1 \end{aligned}$$

$$\begin{aligned} E(Y|X=x) &= \int_x^{x+1} y \cdot dy \\ &= \frac{1}{2} \cdot y^2 \Big|_x^{x+1} = \frac{1}{2} \cdot ([x+1]^2 - x^2) = x + \frac{1}{2} \end{aligned}$$

$$\begin{aligned} E(Y^2|X=x) &= \int_x^{x+1} y^2 \cdot dy \\ &= \frac{1}{3} \cdot y^3 \Big|_x^{x+1} = \frac{1}{3} \cdot ([x+1]^3 - x^3) = x^2 + x + \frac{1}{3} \end{aligned}$$

$$\begin{aligned} V(Y|X=x) &= x^2 + x + \frac{1}{3} - \left(x + \frac{1}{2}\right)^2 \\ &= x^2 + x + \frac{1}{3} - x^2 - x - \frac{1}{4} \\ &= \frac{1}{12} \\ &= \boxed{A} \end{aligned}$$